Some linear Jacobi structures on vector bundles

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Abstract. We study Jacobi structures on the dual bundle A^* to a vector bundle A such that the Jacobi bracket of linear functions is again linear and the Jacobi bracket of a linear function and the constant function 1 is a basic function. We prove that a Lie algebroid structure on A and a 1-cocycle $\phi \in \Gamma(A^*)$ induce a Jacobi structure on A^* satisfying the above conditions. Moreover, we show that this correspondence is a bijection. Finally, we discuss some examples and applications.

Quelques structures de Jacobi linéaires sur des fibrés vectoriels

Résumé. On étudie des structures de Jacobi sur le fibré dual A^* d'un fibré vectoriel A tels que le crochet de Jacobi de fonctions linéaires est à nouveau linéaire et le crochet de Jacobi d'une fonction linéaire et la fonction constante 1 est une fonction basique. On démontre qu'une structure d'algébroïde de Lie sur A et un 1-cocycle $\phi \in \Gamma(A^*)$ induisent une structure de Jacobi sur A^* qui vérifie les conditions antérieures. On voit aussi que cette correspondance est une bijection. On montre finalement quelques exemples et applications.

Version française abrégée

Soit M une variété différentiable et $\pi : A \to M$ un fibré vectoriel sur M.

Un cocycle pour une structure d'algébroïde de Lie sur $\pi : A \to M$ est une section ϕ du fibré dual $\pi^* : A^* \to M$ tel que $\phi[\![\mu, \eta]\!] = \rho(\mu)(\phi(\eta)) - \rho(\eta)(\phi(\mu))$, pour tout $\mu, \eta \in \Gamma(A)$, où $[\![,]\!]$ est le crochet de Lie sur l'espace $\Gamma(A)$ des sections de $\pi : A \to M$ et $\rho : A \to TM$ est l'application ancre (voir [13]). On dénote donc par $\tilde{\mathcal{A}}$ l'ensemble des paires $(([\![,]\!], \rho), \phi)$, où $([\![,]\!], \rho)$ est une structure d'algébroïde de Lie sur $\pi : A \to M$ et $\phi \in \Gamma(A^*)$ un 1-cocycle. D'ailleurs, on dénote par $\mathcal{J}$ l'ensemble des structures de Jacobi (Λ, E) sur A^*, lesquelles satisfont les deux conditions suivantes:

(C1) Le crochet de Jacobi de deux fonctions linéaires est linéaire.

(C2) Le crochet de Jacobi d'une fonction linéaire et la fonction constante 1 est une fonction basique.

On démontre donc, dans cette note, qu'il y a une correspondance bijective $\Psi : \tilde{\mathcal{A}} \to \mathcal{J}$ entre les ensembles $\tilde{\mathcal{A}}$ et $\mathcal{J}$. L'application Ψ est défini par $\Psi(([\![,]\!], \rho), \phi) = (\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$ avec

$$\Lambda_{(A^*,\phi)} = \Lambda_{A^*} + \Delta \wedge \phi^v, \qquad E_{(A^*,\phi)} = -\phi^v,$$

où Λ_{A^*} est le bi-vecteur de Poisson sur A^* induit par la structure d'algébroïde de Lie $([\![,]\!], \rho)$ (voir [2, 3]), Δ est le champ de Liouville sur A^* et ϕ^v est le relèvement vertical de ϕ. Observons que les paires dans $\tilde{\mathcal{A}}$ de la forme $(([\![,]\!], \rho), 0)$ correspondent, à travers Ψ, aux structures de Poisson dans $\mathcal{J}$. Ainsi, comme conséquence, on déduit un résultat démontré dans [2, 3].

Les conditions (C1) et (C2) établies ci-dessus sont naturelles. En fait, on démontre que celles-ci sont vérifiées pour quelques structures de Jacobi, bien connues et importantes, définies sur l'espace

total de quelques fibrés vectoriels. En même temps, la correspondance Ψ nous permet d'obtenir de nouveaux et intéressants exemples de structures de Jacobi. On voit finalement, comme une autre application, qu'une structure d'algébroïde de Lie sur un fibré vectoriel $A \to M$ et un 1-cocycle $\phi \in \Gamma(A^*)$ induisent une structure d'algébroïde de Lie sur le fibré vectoriel $A \times I\!\!R \to M \times I\!\!R$.

1 Jacobi manifolds and Lie algebroids

Let M be a differentiable manifold of dimension n. We will denote by $C^\infty(M, I\!\!R)$ the algebra of C^∞ real-valued functions on M, by $\Omega^1(M)$ the space of 1-forms, by $\mathfrak{X}(M)$ the Lie algebra of vector fields and by $[\, ,]$ the Lie bracket of vector fields.

A *Jacobi structure* on M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda]_{SN} = 2E \wedge \Lambda, \qquad [E, \Lambda]_{SN} = 0. \tag{1}$$

Here $[\ ,\ ]_{SN}$ denotes the Schouten-Nijenhuis bracket ([1, 14]). The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by $\{\bar{f}, \bar{g}\} = \Lambda(d\bar{f}, d\bar{g}) + \bar{f}E(\bar{g}) - \bar{g}E(\bar{f})$, for all $\bar{f}, \bar{g} \in C^\infty(M, I\!\!R)$. Note that

$$\{\bar{f}, \bar{g}\bar{h}\} = \bar{g}\{\bar{f}, \bar{h}\} + \bar{h}\{\bar{f}, \bar{g}\} - \bar{g}\bar{h}\{\bar{f}, 1\}. \tag{2}$$

In fact, the space $C^\infty(M, I\!\!R)$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [8]). Conversely, a structure of local Lie algebra on $C^\infty(M, I\!\!R)$ defines a Jacobi structure on M (see [5, 8]). If the vector field E identically vanishes then (M, Λ) is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz ([10, 11]) (see also [1, 4, 12, 14, 15]).

A *Lie algebroid structure* on a differentiable vector bundle $\pi : A \to M$ is a pair that consists of a Lie algebra structure $[\![\, ,]\!]$ on the space $\Gamma(A)$ of the global cross sections of $\pi : A \to M$ and a homomorphism of vector bundles $\rho : A \to TM$, the *anchor map*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, I\!\!R)$-modules induced by the anchor map then: (i) $\rho : (\Gamma(A), [\![\, ,]\!]) \to (\mathfrak{X}(M), [\, ,])$ is a Lie algebra homomorphism and (ii) for all $\bar{f} \in C^\infty(M, I\!\!R)$ and for all $\mu, \eta \in \Gamma(A)$, one has $[\![\mu, \bar{f}\eta]\!] = \bar{f}[\![\mu, \eta]\!] + (\rho(\mu)(\bar{f}))\eta$ (see [13]).

If $(A, [\![\, ,]\!], \rho)$ is a Lie algebroid over M, one can introduce the Lie algebroid cohomology complex with trivial coefficients (for the explicit definition of this complex we remit to [13]). The space of 1-cochains is $\Gamma(A^*)$, where A^* is the dual bundle to A, and a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi[\![\mu, \eta]\!] = \rho(\mu)(\phi(\eta)) - \rho(\eta)(\phi(\mu)), \text{ for all } \mu, \eta \in \Gamma(A). \tag{3}$$

A Jacobi manifold (M, Λ, E) has an associated Lie algebroid $(T^*M \times I\!\!R, [\![\, ,]\!]_{(\Lambda,E)}, \#_{(\Lambda,E)})$, where T^*M is the cotangent bundle of M and $[\![\, ,]\!]_{(\Lambda,E)}$, $\#_{(\Lambda,E)}$ are defined by

$$\begin{aligned} [\![(\alpha, \bar{f}), (\beta, \bar{g})]\!]_{(\Lambda,E)} &= (\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - d(\Lambda(\alpha, \beta)) + \bar{f}\mathcal{L}_E\beta - \bar{g}\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta), \\ &\quad\ \Lambda(\beta, \alpha) + \#_\Lambda(\alpha)(\bar{g}) - \#_\Lambda(\beta)(\bar{f}) + \bar{f}E(\bar{g}) - \bar{g}E(\bar{f})), \\ \#_{(\Lambda,E)}(\alpha, \bar{f}) &= \#_\Lambda(\alpha) + \bar{f}E, \end{aligned} \tag{4}$$

for $(\alpha, \bar{f}), (\beta, \bar{g}) \in \Omega^1(M) \times C^\infty(M, I\!\!R)$, $\mathcal{L}$ being the Lie derivative operator and $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ the mapping given by $\beta(\#_\Lambda(\alpha)) = \Lambda(\alpha, \beta)$ (see [9]).

In the particular case when (M, Λ) is a Poisson manifold we recover, by projection, the Lie algebroid $(T^*M, [\![\, ,]\!]_\Lambda, \#_\Lambda)$, where $[\![\, ,]\!]_\Lambda$ is the bracket of 1-forms defined by (see [1, 2, 14]):

$$[\![\, ,]\!]_\Lambda : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \quad [\![\alpha, \beta]\!]_\Lambda = \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - d(\Lambda(\alpha, \beta)).$$

2 Some linear Jacobi structures on vector bundles

Let $\pi : A \to M$ be a vector bundle and A^* the dual bundle to A. Suppose that $\pi^* : A^* \to M$ is the canonical projection. If $\mu \in \Gamma(A)$ and $\bar{f} \in C^\infty(M, I\!R)$ then μ determines a linear function on A^* which we will denote by $\tilde{\mu}$ and $f = \bar{f} \circ \pi^*$ is a C^∞ real-valued function on A^* which is basic.

Now, assume that $(A, [\![,]\!], \rho)$ is a Lie algebroid over M. Then A^* admits a Poisson structure Λ_{A^*} such that the Poisson bracket of linear functions is again linear (see [2, 3]). The local expression of Λ_{A^*} is given as follows. Let U be an open coordinate neighbourhood of M with coordinates $(x^1, \dots, x^m)$ and $\{e_i\}_{i=1,\dots,n}$ a local basis of sections of $\pi : A \to M$ in U. Then, $(\pi^*)^{-1}(U)$ is an open coordinate neighbourhood of A^* with coordinates (x^i, μ_j) such that $\mu_j = \tilde{e}_j$, for all j. In these coordinates the structure functions and the components of the anchor map are

$$[\![e_i, e_j]\!] = \sum_{k=1}^{n} c_{ij}^k e_k, \qquad \rho(e_i) = \sum_{l=1}^{m} \rho_i^l \frac{\partial}{\partial x^l}, \quad i, j \in \{1, \dots, n\}, \tag{5}$$

with $c_{ij}^k, \rho_i^l \in C^\infty(U, I\!R)$, and the Poisson structure Λ_{A^*} is given by

$$\Lambda_{A^*} = \sum_{i<j} \sum_{k} c_{ij}^k \mu_k \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial \mu_j} + \sum_{i,l} \rho_i^l \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial x^l}. \tag{6}$$

Next, we will show an extension of the above results for the Jacobi case.

We will denote by Δ the *Liouville vector field* of A^* and by $\phi^v \in \mathfrak{X}(A^*)$ the *vertical lift* of $\phi \in \Gamma(A^*)$. Note that if (x^i, μ_j) are fibred coordinates on A^* as above and $\phi = \sum_{i=1}^{n} \phi_i e^i$, with $\phi_i \in C^\infty(U, I\!R)$ and $\{e^i\}$ the dual basis of $\{e_i\}$, then

$$\Delta = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial \mu_i}, \qquad \phi^v = \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial \mu_i}. \tag{7}$$

Thus, using (1), (3), (5), (6) and (7), we deduce

Theorem 1 *Let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ a 1-cocycle. Then, there is a unique Jacobi structure $(\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$ on A^* with Jacobi bracket $\{\,,\,\}_{(A^*,\phi)}$ satisfying*

$$\{\tilde{\mu}, \tilde{\eta}\}_{(A^*,\phi)} = \widetilde{[\![\mu, \eta]\!]}, \quad \{\tilde{\mu}, \bar{f} \circ \pi^*\}_{(A^*,\phi)} = (\rho(\mu)(\bar{f}) + \phi(\mu)\bar{f}) \circ \pi^*, \quad \{\bar{f} \circ \pi^*, \bar{g} \circ \pi^*\}_{(A^*,\phi)} = 0,$$

for $\mu, \eta \in \Gamma(A)$ and $\bar{f}, \bar{g} \in C^\infty(M, I\!R)$. The Jacobi structure is given by

$$\Lambda_{(A^*,\phi)} = \Lambda_{A^*} + \Delta \wedge \phi^v, \qquad E_{(A^*,\phi)} = -\phi^v.$$

Now, we will prove a converse of Theorem 1.

Theorem 2 *Let $\pi : A \to M$ be a vector bundle over M and let (Λ, E) be a Jacobi structure on the dual bundle A^* satisfying:*

(C1) The Jacobi bracket of linear functions is again linear.

(C2) The Jacobi bracket of a linear function and the constant function 1 is a basic function.

Then, there is a Lie algebroid structure on $\pi : A \to M$ and a 1-cocycle $\phi \in \Gamma(A^*)$ such that $\Lambda = \Lambda_{(A^*,\phi)}$ and $E = E_{(A^*,\phi)}$.

Proof: Denote by $\{\ ,\ \}$ the Jacobi bracket on A^* induced by the Jacobi structure (Λ, E) and suppose that $\mu, \eta \in \Gamma(A)$ and that $\bar{f}, \bar{g} \in C^\infty(M, I\!\!R)$. If $\pi^* : A^* \to M$ is the canonical projection, the function $\{(\bar{f} \circ \pi^*)\tilde{\mu}, 1\} = \{\widetilde{\bar{f}\mu}, 1\}$ is basic. Thus, from (2) and *(C2)*, we have that

$$\{\bar{f} \circ \pi^*, 1\} = 0. \tag{8}$$

On the other hand, the function $\{\tilde{\mu}, (\bar{f} \circ \pi^*)\tilde{\eta}\} = \{\tilde{\mu}, \widetilde{\bar{f}\eta}\}$ is linear. Therefore, from (2), *(C1)* and *(C2)*, we obtain that the function $\{\tilde{\mu}, \bar{f} \circ \pi^*\}$ is basic. Consequently, the Jacobi bracket of a linear function and a basic function is a basic function. In particular, $\{\bar{f} \circ \pi^*, (\bar{g} \circ \pi^*)\tilde{\mu}\} = \{\bar{f} \circ \pi^*, \widetilde{\bar{g}\mu}\}$ is basic. This implies that (see (2) and (8))

$$\{\bar{f} \circ \pi^*, \bar{g} \circ \pi^*\} = 0. \tag{9}$$

Now, we define the section $[\![\mu, \eta]\!]$ of the vector bundle $\pi : A \to M$ and the C^∞ real-valued functions on M, $\phi(\mu)$ and $\rho(\mu)(\bar{f})$, which are characterized by the following relations

$$\widetilde{[\![\mu, \eta]\!]} = \{\tilde{\mu}, \tilde{\eta}\}, \quad \phi(\mu) \circ \pi^* = \{\tilde{\mu}, 1\}, \quad \rho(\mu)(\bar{f}) \circ \pi^* = \{\tilde{\mu}, \bar{f} \circ \pi^*\} - (\bar{f} \circ \pi^*)\{\tilde{\mu}, 1\}. \tag{10}$$

From (2), (8), (9) and (10), we deduce that ϕ can be considered as a $C^\infty(M, I\!\!R)$-linear map $\phi : \Gamma(A) \to C^\infty(M, I\!\!R)$ (that is, $\phi \in \Gamma(A^*)$) and that ρ can be considered as a $C^\infty(M, I\!\!R)$-linear map $\rho : \Gamma(A) \to \mathfrak{X}(M)$. Moreover, using (2), (3), (10) and the fact that $\{\ ,\ \}$ is the Jacobi bracket of a Jacobi structure (see Section 1), it follows that the triple $(A, [\![,]\!], \rho)$ is a Lie algebroid over M and that $\phi \in \Gamma(A^*)$ is a 1-cocycle. Finally, by (9), (10) and Theorem 1, we conclude that $(\Lambda, E) = (\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$. $\boxed{QED}$

Remark 1 That condition *(C1)* does not necessarily imply condition *(C2)* is illustrated by the following simple example. Let M be a single point and $A^* = I\!\!R^2$ endowed with the Jacobi structure (Λ, E), where $\Lambda = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and $E = x\frac{\partial}{\partial x}$. It is easy to prove that the Jacobi bracket satisfies *(C1)* but not *(C2)*.

Let M be a differentiable manifold and $\pi : A \to M$ a vector bundle. Denote by $\tilde{\mathcal{A}}$ and $\mathcal{J}$ the following sets. $\tilde{\mathcal{A}}$ is the set of the pairs $(([\![,]\!], \rho), \phi)$, where $([\![,]\!], \rho)$ is a Lie algebroid structure on $\pi : A \to M$ and $\phi \in \Gamma(A^*)$ is a 1-cocycle. $\mathcal{J}$ is the set of the Jacobi structures (Λ, E) on A^* which satisfy the conditions *(C1)* and *(C2)* (see Theorem 2).

Then, using Theorems 1 and 2, we obtain

Theorem 3 *The mapping $\Psi : \tilde{\mathcal{A}} \to \mathcal{J}$ between the sets $\tilde{\mathcal{A}}$ and $\mathcal{J}$ given by*

$$\Psi(([\![,]\!], \rho), \phi) = (\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$$

is a bijection.

Note that $\Psi(\mathcal{A}) = \mathcal{P}$, where $\mathcal{P}$ is the subset of the Jacobi structures of $\mathcal{J}$ which are Poisson and $\mathcal{A}$ is the subset of $\tilde{\mathcal{A}}$ of the pairs of the form $(([\![,]\!], \rho), 0)$, that is, $\mathcal{A}$ is the set of the Lie algebroid structures on $\pi : A \to M$. Therefore, from Theorem 3, we deduce a well known result (see [2, 3]): the mapping Ψ induces a bijection between the sets $\mathcal{A}$ and $\mathcal{P}$.

3 Examples and applications

In this section we will present some examples and applications of the results obtained in Section 2.

1.- Let $(\mathfrak{g}, [\,,\,])$ be a real Lie algebra of dimension n. Then, $\mathfrak{g}$ is a Lie algebroid over a point. The resultant Poisson structure $\Lambda_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is the well known *Lie-Poisson structure* (see (6)). Thus, if $\phi \in \mathfrak{g}^*$ is a 1-cocycle then, using Theorem 1, we deduce that the pair $(\Lambda_{(\mathfrak{g}^*,\phi)}, E_{(\mathfrak{g}^*,\phi)})$ is a Jacobi structure on $\mathfrak{g}^*$, where

$$\Lambda_{(\mathfrak{g}^*,\phi)} = \Lambda_{\mathfrak{g}^*} + R \wedge C_\phi, \quad E_{(\mathfrak{g}^*,\phi)} = -C_\phi,$$

R is the radial vector field on $\mathfrak{g}^*$ and C_ϕ is the constant vector field on $\mathfrak{g}^*$ induced by $\phi \in \mathfrak{g}^*$.

2.- Let $(TM, [\,,], Id)$ be the trivial Lie algebroid. In this case, the Poisson structure Λ_{T^*M} on T^*M is the *canonical symplectic structure*. Therefore, if ϕ is a closed 1-form on M, then the pair

$$\Lambda_{(T^*M,\phi)} = \Lambda_{T^*M} + \Delta \wedge \phi^v, \quad E_{(T^*M,\phi)} = -\phi^v,$$

is a Jacobi structure on T^*M. Furthermore, we can prove that the map $\#_{\Lambda_{(T^*M,\phi)}} : \Omega^1(T^*M) \to \mathfrak{X}(T^*M)$ is an isomorphism and consequently, using the results of [5, 8] (see also [4]), it follows that $(\Lambda_{(T^*M,\phi)}, E_{(T^*M,\phi)})$ is a *locally conformal symplectic structure*.

3.- Let (M, Λ) be a Poisson manifold and $(T^*M, [\![\,,]\!]_\Lambda, \#_\Lambda)$ the associated cotangent Lie algebroid (see Section 1). The induced Poisson structure on TM is the *complete lift* Λ^c to TM of Λ (see [3]). Thus, if $X \in \mathfrak{X}(M) = \Gamma(TM)$ is a 1-cocycle, that is, X is a Poisson infinitesimal automorphism ($\mathcal{L}_X\Lambda = 0$), we deduce that

$$\Lambda_{(TM,X)} = \Lambda^c + \Delta \wedge X^v, \quad E_{(TM,X)} = -X^v,$$

is a Jacobi structure on TM.

4.- The triple $(TM \times I\!R, [\ ,\ ], \pi)$ is a Lie algebroid over M, where $\pi : TM \times I\!R \to TM$ is the canonical projection over the first factor and $[\ ,\ ]$ is the bracket given by

$$[(X, \bar{f}), (Y, \bar{g})] = ([X, Y], X(\bar{g}) - Y(\bar{f})), \text{ for } (X, \bar{f}), (Y, \bar{g}) \in \mathfrak{X}(M) \times C^\infty(M, I\!R). \tag{11}$$

In this case, the Poisson structure $\Lambda_{T^*M\times I\!R}$ on $T^*M \times I\!R$ is just the *canonical cosymplectic structure* of $T^*M \times I\!R$, that is, $\Lambda_{T^*M\times I\!R} = \Lambda_{T^*M}$. Now, it is easy to prove that $\phi = (0, -1) \in \Omega^1(M) \times C^\infty(M, I\!R) = \Gamma(T^*M \times I\!R)$ is a 1-cocycle (see (3) and (11)). Moreover, using Theorem 1, we have that the Jacobi structure $(\Lambda_{(T^*M\times I\!R,\phi)}, E_{(T^*M\times I\!R,\phi)})$ on $T^*M \times I\!R$ is the one defined by the *canonical contact 1-form* η_M. We recall that η_M is the 1-form on $T^*M \times I\!R$ given by $\eta_M = dt + \lambda_M$, λ_M being the *Liouville 1-form* of T^*M (see [12]).

5.- Let (M, Λ, E) be a Jacobi manifold and $(T^*M \times I\!R, [\![\,,]\!]_{(\Lambda,E)}, \#_{(\Lambda,E)})$ the associated Lie algebroid (see Section 1). From (1), (3) and (4), it follows that $\phi = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, I\!R) = \Gamma(TM \times I\!R)$ is a 1-cocycle. On the other hand, a long computation, using (4), (6), (7) and Theorem 1, shows that

$$\Lambda_{(TM\times I\!R,\phi)} = \Lambda^c + \frac{\partial}{\partial t} \wedge E^c - t\Big(\Lambda^v + \frac{\partial}{\partial t} \wedge E^v\Big), \qquad E_{(TM\times I\!R,\phi)} = E^v,$$

where Λ^c (resp. Λ^v) is the complete (resp. vertical) lift to TM of Λ and E^c (resp. E^v) is the complete (resp. vertical) lift to TM of E. We remark that in [6] the authors characterize the conformal infinitesimal automorphisms of (M, Λ, E) as Legendre-Lagrangian submanifolds of the Jacobi manifold $(TM \times I\!R, \Lambda_{(TM\times I\!R,\phi)}, E_{(TM\times I\!R,\phi)})$.

6.- Let $(A, [\![\,,]\!], \rho)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ a 1-cocycle. Denote by $\hat{\Lambda}_{A^*\times I\!R}$ the Poissonization of the Jacobi structure $(\Lambda_{(A^*,\phi)}, E_{(A^*,\phi)})$, that is, $\hat{\Lambda}_{A^*\times I\!R}$ is the Poisson structure

on $\hat{A}^* = A^* \times I\!R$ given by (see [5, 11])

$$\hat{\Lambda}_{A^* \times I\!R} = e^{-t}\Big(\Lambda_{(A^*,\phi)} + \frac{\partial}{\partial t} \wedge E_{(A^*,\phi)}\Big). \tag{12}$$

$\hat{A}^*$ is the total space of a vector bundle over $M \times I\!R$ and, from (12), we obtain that the Poisson bracket of two linear functions on $\hat{A}^*$ is again linear. This implies that the dual vector bundle $\hat{A} = A \times I\!R \to M \times I\!R$ admits a Lie algebroid structure $([\![,]\!]\hat{}, \hat{\rho})$. Note that the space $\Gamma(\hat{A})$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we deduce that (see (10) and (12))

$$[\![\hat{\mu}, \hat{\eta}]\!]\hat{} = e^{-t}\Big([\![\hat{\mu}, \hat{\eta}]\!] + \phi(\hat{\mu})(\frac{d\hat{\eta}}{dt} - \hat{\eta}) - \phi(\hat{\eta})(\frac{d\hat{\mu}}{dt} - \hat{\mu})\Big), \quad \hat{\rho}(\hat{\mu}) = e^{-t}\Big(\rho(\hat{\mu}) + \phi(\hat{\mu})\frac{\partial}{\partial t}\Big),$$

for all $\hat{\mu}, \hat{\eta} \in \Gamma(\hat{A})$, where $\frac{d\hat{\mu}}{dt}$ (resp. $\frac{d\hat{\eta}}{dt}$) is the derivative of $\hat{\mu}$ (resp. $\hat{\eta}$) with respect to the time. Note that if $t \in I\!R$ then the sections $\hat{\mu}$ and $\hat{\eta}$ induce, in a natural way, two sections $\hat{\mu}_t$ and $\hat{\eta}_t$ of $A \to M$ and that $[\![\hat{\mu}, \hat{\eta}]\!]$ is the time-dependent section of $A \to M$ given by $[\![\hat{\mu}, \hat{\eta}]\!](x,t) = [\![\hat{\mu}_t, \hat{\eta}_t]\!](x)$, for all $(x,t) \in M \times I\!R$.

The construction of the Lie algebroid $(\hat{A}, [\![,]\!]\hat{}, \hat{\rho})$ from the Lie algebroid $(A, [\![,]\!], \rho)$ and the cocycle ϕ plays an important role in [7].

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